Mathematics 221 Fall 2003

Exam 5 / Final Exam

December 16, 2003

____
Name

Technology used: ____

Directions: Be sure to include in-line citations every time you use technology. Include a careful sketch of any graph obtained by technology in solving a problem. **Only write on one side of each page.**

Exam 5: Questions on Chapter 13.

Do any four (4) of these problems.

Do not use the fundamental theorem of line integrals unless you first show the vector field is conservative.

1. (25 points) Given the vector field $\vec{\mathbf{F}} = e^{xyz}\widehat{\mathbf{i}} + \sin(x-y)\widehat{\mathbf{j}} - \frac{xy}{z}\widehat{\mathbf{k}}$, compute both of the following

   (a) $\operatorname{div}\left(\vec{\mathbf{F}}\right)$

   (b) $\operatorname{curl}\left(\vec{\mathbf{F}}\right)$

2. (25 points) Find the work done by the force field $\vec{\mathbf{F}}(x,y) = x^2\,\mathbf{i} + xy\,\mathbf{j}$ on a particle that moves once around the circle $x^2+y^2=4$ oriented in the **counter-clockwise** direction.

3. (25 points) The force exerted on an electric charge at the origin on a charged particle at a point (x,y,z) with position vector $\overrightarrow{r} =< x,y,z >$ and $\|\vec{r}\| = \sqrt{x^2+y^2+z^2}$ is

$$\mathbf{F}\left(\overrightarrow{r}\right) = K\frac{\overrightarrow{r}}{\|\vec{r}\|^3}$$

   where K is a constant. Find the work done as the particle moves along a straight line from $(2,0,0)$ to $(2,1,5)$.

4. (25 points) Show that the vector field $\mathbf{F}(x,y) =< 2xy+\sin y, x^2+x\cos y >$ is conservative and use the fundamental theorem of line integrals to compute $\int_C \mathbf{F}\cdot\mathbf{d}\overrightarrow{r}$ where C is any curve that starts at $(-3,-3)$ and ends at $(1,1)$.

5. (25 points) Given the vector field $\vec{\mathbf{F}}(x,y) = \left\langle -\frac{y}{2}, \frac{x}{2}\right\rangle$,

   (a) Use Green's Theorem to show that if and C is a simple closed curve enclosing a simply-connected region D, then the area of region D is given by

$$A(D) = \oint_C \vec{\mathbf{F}}\cdot\, d\vec{\mathbf{r}} = \oint_C \left\langle -\frac{y}{2}, \frac{x}{2}\right\rangle \cdot d\vec{\mathbf{r}}.$$

   (b) Use the parametrization $x = a\cos(\theta)$, $y = b\sin(\theta)$ and the formula in part (a.) to find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

6. (25 points) Compute $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $\vec{\mathbf{F}} = \left\langle 3y - e^{\sin(x)}, 7x + \sqrt{y^4+1} \right\rangle$ and C is the circle $x^2 + y^2 = 9$.

Final Examination (Cumulative)

Do any five (5) of these problems

1. (20 points) Do **both** of the following.

   (a) Find all points the line $x = 1 - t$, $y = 3t$, $z = 1 + t$ has in common with the sphere $2x^2 + 2y^2 + 2z^2 + x + y + z = 9$.

   (b) Find an equation of the plane that contains the line $\frac{x-5}{-1} = \frac{y+2}{4} = \frac{z-3}{2}$ and passes through the point $(1, 2, 3)$.

2. (20 points) Draw a reasonable sketch of the quadric surface given by $z = x^2 + xy + y^2 - 4x - 4y + 2$. [Hint: Since it is a quadric surface, the second-derivative test will tell you what type of surface it is.]

3. (20 points) The temperature $T(x, y, z)$ at the point $P(x, y, z)$ of a metal ball whose center is located at the origin is proportional to the distance from P to the center of the ball. The temperature at the point $(1, 2, 2)$ is $120°$. Find the directional derivative of T at the point $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.

4. (20 points) Do **one** (1) of the following.

   (a) The base of an aquarium with volume $540\,\text{cm}^3$ is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of a topless aquarium. Be sure to explain how you know your answer is a minimum.

   (b) Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the sphere $x^2 + y^2 + z^2 = 9$. Be sure to explain how you know your answer is a maximum.

5. (20 points) Do **one** (1) of the following.

   (a) Find the volume of the solid inside both the cylinder $x^2 + y^2 = 4$, and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$.

   (b) Use triple integrals to find the 'hyper-volume' of a 4 - dimensional hypersphere $x^2 + y^2 + z^2 + w^2 = R^2$. Here R is a constant and the result of projecting ("smashing") the hypersphere into xyz-space is the three dimensional sphere $x^2 + y^2 + z^2 = R^2$.

6. (20 points) Do **one** (1) of the following.

   (a) Rewrite the iterated triple integral below in the given orders.

   $$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-z} f(x, y, z)\; dy\, dz\, dx$$

   i. $\iiint f(x, y, z)\; dz\, dy\, dx$

   (b) Carefully describe the solid that is the domain of integration. **Do not evaluate.**

   $$\int_{\pi/6}^{\pi/2} \int_{\csc\phi}^{2\csc\phi} \int_0^{2\pi} \rho \sin^2\phi \, d\theta\, d\rho\, d\phi$$

7. (20 points) Do **one** (1) of the following.

(a) Draw the region R in the xy-plane that corresponds to the square S in the uv-plane with corners $(0,0)$, $(0,1)$,$(1,0)$,$(1,1)$ for the change of variables $x = u\,\cos(v)$, $y = u\,\sin(v)$ [There are only three corners] and use the Jacobian to find the area of region R.

(b) Make the change of variables $u = x + 2y$, $v = x - y$ and evaluate

$$\iint\limits_R \frac{x+2y}{\cos(x-y)}\,dA$$

where R is the parallelogram bounded by the lines $y = x$, $y = x - 1$, $x + 2y = 0$, $x + 2y = 2$.